

# Non-equilibrium equalities with unital quantum channels

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A general tool for description of open quantum systems is given by the formalism of quantum operations. Most important of them are trace-preserving maps also known as quantum channels. We discuss those conditions on quantum channels under which the Jarzynski equality and related fluctuation theorems hold. It is essential that the representing quantum channel be unital. Under the mentioned condition, we first derive the corresponding Jarzynski equality. For bistochastic map and its adjoint, we further formulate a theorem of Tasaki–Crooks type. In the context of unital channels, some notes on heat transfer between two quantum systems are given. We also consider the case of a finite system operated by an external agent with a feedback control. When unital channels are applied at the first stage and, for a mutual-information form, at the further ones, we obtain quantum Jarzynski–Sagawa–Ueda relations. These are extension of the previously given results to unital quantum operations.

Keywords: exact results

## I. INTRODUCTION

Complex systems away from thermal equilibrium are one of the principal issues of statistical physics. Growing interest to thermodynamic of systems at the nanoscale has also stimulated a more detailed study of statistical fluctuations [1]. A particular phenomenon, the so-called "return to equilibrium", was rigorously analyzed [2, 3]. In [4], adiabatic theorems and their connection with the zero law of thermodynamics are reviewed. Further, the role of computer simulations in analyzing non-equilibrium processes will certainly increase. Many efficient schemes for simulation of complicated biomolecules and colloidal particles were developed [5]. Due to Jarzynski [6, 7], novel advances have been achieved for thermodynamic systems driven out from the equilibrium by external forces [1, 8]. If these forces are varied in line with a specified protocol, then some exact relations can be derived. The first of exact non-equilibrium relations is now referred to as the Jarzynski equality. Results of such a kind are significant in own rights as well as for extending the scope of computer simulations [9].

Let us consider a thermally insulated system, which is acted upon by a time-dependent external field. For any quasi-static process, the work performed on the system is equal to the difference between the final and initial free energies. For a non-equilibrium process, averaged total work will exceed this difference due to the second law of thermodynamics. Instead of inequalities, Jarzynski gave the equality connecting non-equilibrium quantities with the equilibrium free energies [6, 7]. In [10], he studied Clausius–Duhem processes via averaging over the ensemble of microscopic realizations and again obtained his non-equilibrium equality. As Crooks showed [11], Jarzynski's equality follows from the assumption that the system dynamics is Markovian and microscopically reversible. These conditions are commonly used in computer simulations. In [12], Crooks further derived a related fluctuation theorem of own significance. Quantum counterparts of both the classical formulations were given by Tasaki [13]. Numerous aspects of Jarzynski's equality and some related results have been addressed in [14–17]. Quantum non-equilibrium work relations are still the subject of active research [18–21].

Concerning the experimental verification of new relations, a principal difference exists between the classical and quantum regimes [16]. The classical formulations are actually much easier to validate. Original Jarzynski's equality and Crooks' theorem has been tested in experiments with individual biomolecules [22, 23], macroscopic oscillator [24] and electronic system [25]. The latter was also studied numerically [26]. The writers of [27] demonstrated feedback manipulations with a Brownian particle. This allowed to test the corresponding version of Jarzynski's equality given in [28] (for more details, see [29]). At the same time, an experimental validation of the quantum fluctuation relations is still lacking. In [30–32], some experimental setups to verify the quantum relations with use of current technology were proposed (for a discussion, see also sect. VI of [16]). Note that quantum proposals deal with single particles under going unitary evolution. On the other hand, quantum systems are very sensitive to noise. To understand the issue properly, it should be considered from as many aspects as possible. In particular, theoretical studies of non-equilibrium fluctuations in open quantum systems may be interesting for possible future experiments.

Ways to derive Jarzynski's equality are often based on description within an infinitesimal time scale. In the classical case, we can use master equation [7, 33], some forms of deterministic dynamics [34, 35] or Markovian dynamics [11, 12]. In the quantum regime, many approaches were done with somewhat particular assumptions such as description by Schrödinger [20] or master equation [36], unitary evolution [13, 19], or the time-reversal symmetry [21]. An extension of the Hamiltonian approach to arbitrary quantum systems has been considered in [18]. Notably, in Jarzynski's equality we actually deal with quantities only at the initial and final points, without explicit reference to the passage of time.

Hence, we may be interested in obtaining Jarzynski's equality with only discrete state changes. Certainly, reversible unitary transformations form a very special class of possible state changes. The formalism of quantum operations is one of basic tools in studying dynamics of open quantum systems. This formalism is especially well adapted to describe discrete state changes. By such a property, quantum operations are widely used in quantum information theory [37]. Deterministic processes are represented by trace-preserving maps known as quantum channels.

The aim of this paper is to study quantum versions of the Jarzynski equality and related results with use of quantum operation techniques. The paper is organized as follows. In Section II, the preliminary material is reviewed. Basic notions of quantum operation techniques are recalled. Joint probability distribution for measurements statistics is discussed as well. Hence, the averaging rule for considered topics is obtained. In Section III, we obtain Jarzynski's equality under condition that the considered process is represented by a unital quantum channel. Further, we formulate a fluctuation theorem of Tasaki–Crooks type for bistochastic map and its adjoint. We also discuss a heat transfer between two quantum systems. Initially, the combined system is prepared in the product state of two particular densities of a special form. We end Section III with a short comments about physical relevance of unital channels and their distinctions from unitary ones. In Section IV, some equalities with unital channels in the case of feedback control are obtained. First, we examine the case of error-free feedback control. Second, we extend the formulation to a feedback control with classical errors. In Section V, we conclude the paper with a summary of results.

## II. DEFINITIONS AND NOTATION

In this section, the required material is presented. First, we briefly recall basic notions of the formalism of quantum operations. Second, we describe a general form of the averaging procedure with a joint probability distribution.

### A. States, operators, and quantum channels

Let  $\mathcal{L}(\mathcal{H})$  denote the space of linear operators on finite-dimensional Hilbert space  $\mathcal{H}$ . By  $\mathcal{L}_{s.a.}(\mathcal{H})$  and  $\mathcal{L}_+(\mathcal{H})$ , we respectively mean the real space of Hermitian operators and the set of positive ones. For arbitrary  $\mathbf{X}, \mathbf{Y} \in \mathcal{L}(\mathcal{H})$ , we define their Hilbert–Schmidt inner product by [38]

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{\text{hs}} := \text{Tr}(\mathbf{X}^\dagger \mathbf{Y}) . \quad (1)$$

We now consider a linear map  $\Phi : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$  that takes elements of  $\mathcal{L}(\mathcal{H}_A)$  to elements of  $\mathcal{L}(\mathcal{H}_B)$ . To describe a physical process, this map must be completely positive [37, 39]. Let  $\text{id}_R$  be the identity map on  $\mathcal{L}(\mathcal{H}_R)$ , where the space  $\mathcal{H}_R$  is assigned to a reference system. The complete positivity implies that  $\Phi \otimes \text{id}_R$  transforms each positive operator into a positive operator again for each dimension of the extended space. Any completely positive map can be written in the operator-sum representation. For all  $\mathbf{X} \in \mathcal{L}(\mathcal{H}_A)$ , we have

$$\Phi(\mathbf{X}) = \sum_{\mu} \mathbf{K}_{\mu} \mathbf{X} \mathbf{K}_{\mu}^\dagger . \quad (2)$$

Here, the Kraus operators  $\mathbf{K}_{\mu}$  map the input space  $\mathcal{H}_A$  to the output space  $\mathcal{H}_B$ . To each linear map  $\Phi : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$ , we assign its adjoint  $\Phi^\dagger : \mathcal{L}(\mathcal{H}_B) \rightarrow \mathcal{L}(\mathcal{H}_A)$  by the formula [38]

$$\langle \Phi(\mathbf{X}), \mathbf{Y} \rangle_{\text{hs}} = \langle \mathbf{X}, \Phi^\dagger(\mathbf{Y}) \rangle_{\text{hs}} , \quad (3)$$

which holds for all  $\mathbf{X} \in \mathcal{L}(\mathcal{H}_A)$  and  $\mathbf{Y} \in \mathcal{L}(\mathcal{H}_B)$ . For the linear map (2), its adjoint is represented as

$$\Phi^\dagger(\mathbf{Y}) = \sum_{\mu} \mathbf{K}_{\mu}^\dagger \mathbf{Y} \mathbf{K}_{\mu} . \quad (4)$$

The state of an open quantum system is described by a density matrix  $\boldsymbol{\rho} \in \mathcal{L}_+(\mathcal{H})$  normalized as  $\text{Tr}(\boldsymbol{\rho}) = 1$ . The input density matrix  $\boldsymbol{\rho}_A$  is mapped to the output  $\Phi(\boldsymbol{\rho}_A) \in \mathcal{L}_+(\mathcal{H}_B)$ . To be consistent with probabilistic interpretation, the map  $\Phi$  should obey the condition [37]

$$\sum_{\mu} \mathbf{K}_{\mu}^\dagger \mathbf{K}_{\mu} \leq \mathbb{1}_A , \quad (5)$$

where  $\mathbb{1}_A$  denotes the identity operator on  $\mathcal{H}_A$ . By quantum operations, we mean maps of the form (2) under the restriction (5). Deterministic processes are described by trace-preserving operations, for which the inequality (5) is saturated and, herewith,  $\text{Tr}(\Phi(\boldsymbol{\rho}_A)) = 1$ . These maps are usually referred to as quantum channels [39]. Most

familiar changes of quantum states are represented by unitary transformations of the Hilbert space. In this case, the quantum channel has a unique Kraus operator. To saturate (5), this operator is inevitably unitary, i.e.  $K^\dagger K = \mathbb{1}$ . Non-trace-preserving quantum operations are also used in quantum information [37]. To get the output density matrix for probabilistic operations, we rescale the output as

$$\rho_B := \text{Tr}(\Phi(\rho_A))^{-1} \Phi(\rho_A) . \quad (6)$$

Except for trace-preserving maps, the denominator in (6) generally depends on the input  $\rho_A$ . On the other hand, the macroscopic dynamics is deterministic. For these reasons, we further focus an attention on quantum channels. Two linear maps  $\Phi$  and  $\Psi$  can be composed to obtain another linear map  $\Psi \circ \Phi$  such that

$$(\Psi \circ \Phi)(X) := \Psi(\Phi(X)) , \quad (7)$$

for all  $X \in \mathcal{L}(\mathcal{H}_A)$ . Its operator-sum representation directly follows from the representations of  $\Phi$  and  $\Psi$ . The composition of two completely positive maps is completely positive as well. Hence, such maps form some set with a semigroup structure. The same observation pertains to quantum channels. An essential interest to dynamical semigroups was inspired in studying physical issues such as master equations, quantum noises, quantum communication channels, and so on [40]. Let us consider the operator sum

$$\Phi(\mathbb{1}_A) = \sum_{\mu} K_{\mu} K_{\mu}^{\dagger} . \quad (8)$$

Assume that the operator (8) is a multiple of  $\mathbb{1}_B$ . For trace-preserving maps, this condition gives

$$\Phi(\mathbb{1}_A) = \frac{d_A}{d_B} \mathbb{1}_B . \quad (9)$$

Here, the integers  $d_A = \dim(\mathcal{H}_A)$  and  $d_B = \dim(\mathcal{H}_B)$  are dimensionalities of  $\mathcal{H}_A$  and  $\mathcal{H}_B$ . When the input and output spaces are of the same dimensionality, the formula (9) gives  $\Phi(\mathbb{1}_A) = \mathbb{1}_B$ . The latter is usually expressed as the map  $\Phi$  being unital [37]. For the simplest system, i.e. quantum bit, the depolarizing and phase damping channels are both unital, whereas the amplitude damping channel is not [37]. Note that the depolarizing channel, which represents a decohering qubit, has interesting entropic characteristics [41]. Unital trace-preserving maps are often referred to as bistochastic [39]. It is easy to check that the adjoint of bistochastic map is bistochastic as well. In the following, we derive a useful statement about those trace-preserving maps that satisfy the condition (9).

## B. Joint probability distribution and averaging rule

The basic idea of statistical physics is to represent a macroscopic situation of interest by an ensemble of its microscopic realizations. For a quantum version of Jarzynski's equality, the corresponding framework was explicitly developed by Tasaki [13]. It is convenient to pose an approach initially for arbitrary observables  $A \in \mathcal{L}_{s.a.}(\mathcal{H}_A)$  and  $B \in \mathcal{L}_{s.a.}(\mathcal{H}_B)$ . Their spectral decompositions are expressed as

$$A = \sum_i a_i |a_i\rangle\langle a_i| , \quad (10)$$

$$B = \sum_j b_j |b_j\rangle\langle b_j| . \quad (11)$$

In both the decompositions, the eigenvalues are assumed to be taken according to their multiplicity. In this regard, we treat  $a_i$  and  $b_j$  as the labels for vectors of the orthonormal bases  $\{|a_i\rangle\}$  and  $\{|b_j\rangle\}$ . Suppose the system evolution is represented by quantum channel  $\Phi$ . If the input state is described by eigenstate  $|a_i\rangle$ , then the channel output is  $\Phi(|a_i\rangle\langle a_i|)$ . Then the probability of being the state  $|b_j\rangle$  is calculated as

$$p(b_j|a_i) = \langle b_j | \Phi(|a_i\rangle\langle a_i|) | b_j \rangle . \quad (12)$$

This quantity is the conditional probability of the outcome  $b_j$  given that the input state was  $|a_i\rangle$ . Due to the preservation of the trace, we then obtain

$$\sum_j p(b_j|a_i) = \text{Tr}(\Phi(|a_i\rangle\langle a_i|)) = 1 . \quad (13)$$

Thus, the standard requirement on conditional probabilities is satisfied with any quantum channel. Further, we suppose that the input density matrix  $\rho_A$  has the form

$$\rho_A = \sum_i p(a_i) |a_i\rangle \langle a_i| , \quad (14)$$

where  $\sum_i p(a_i) = 1$ . The operator (14) can be rewritten as a function of the observable  $A$ . Due to the Bayes rule, one defines the joint probability distribution such that

$$p(a_i, b_j) = p(a_i) p(b_j | a_i) . \quad (15)$$

This is the probability that we find the system in  $i$ -th eigenstate of  $A$  at the input and in  $j$ -th eigenstate of  $B$  at the output. Consider a function  $f(a, b)$  of two eigenvalues. Extending Tasaki's approach [13], we define an average

$$\langle f(a, b) \rangle := \sum_{ij} p(a_i, b_j) f(a_i, b_j) . \quad (16)$$

Here, angular brackets in the left-hand side signify averaging over the ensemble of possible pairs of measurement outcomes. Between these brackets, we will usually omit labels of the involved variables. In Section IV, however, we will consider enough complicated protocols. There, the labels will be all indicated for clearness.

In general, the average (16) does not pertain to quantum-mechanical expectation values. Namely, the right-hand side of equation (16) corresponds to specific physical meaning [13]. In two simplest cases, however, the average (16) coincides with the quantum-mechanical expectation value. Let  $a \mapsto g(a)$  be some well-defined function. Due to (13) and (14), we directly obtain

$$\langle g(a) \rangle = \sum_i g(a_i) p(a_i) = \sum_i g(a_i) \langle a_i | \rho_A | a_i \rangle = \text{Tr}(g(A) \rho_A) . \quad (17)$$

Using the map linearity and the formulas (12) and (15), we also write

$$\langle g(b) \rangle = \sum_j g(b_j) \sum_i p(a_i) \langle b_j | \Phi(|a_i\rangle \langle a_i|) | b_j \rangle = \text{Tr}(g(B) \Phi(\rho_A)) . \quad (18)$$

### III. RELATIONS OF JARZYNSKI AND TASAKI-CROOKS TYPES

In this section, we obtain some results connected with Jarzynski's equality and the Tasaki-Crooks fluctuation theorem. Their derivation will mainly be based on the condition that the representing quantum channel is unital. Further, we consider a heat transfer between two quantum systems.

#### A. Jarzynski's equality with unital quantum channels

Before obtaining Jarzynski's equality, we will formulate a mathematical result in more abstract form. We consider the case, in which the input density matrix is expressed as

$$\varrho_A(\alpha) := \text{Tr}(e^{-\alpha A})^{-1} e^{-\alpha A} . \quad (19)$$

Functional form of such a kind is related to the state of thermal equilibrium in the Gibbs canonical ensemble. The following exact relation could be applied beyond the context of Jarzynski's equality.

**Proposition 1** *Let  $A \in \mathcal{L}_{s.a.}(\mathcal{H}_A)$ ,  $B \in \mathcal{L}_{s.a.}(\mathcal{H}_B)$ , and let  $\alpha$  and  $\beta$  be real numbers. Suppose the input state is described by density matrix (19). If the quantum channel  $\Phi : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B)$  satisfies the condition (9), then*

$$\langle \exp(\alpha a - \beta b) \rangle = \frac{d_A}{d_B} \frac{\text{Tr}(e^{-\beta B})}{\text{Tr}(e^{-\alpha A})} . \quad (20)$$

**Proof.** Using the linearity of the map  $\Phi$  and the condition (9), we obtain

$$\sum_i p(b_j | a_i) = \langle b_j | \sum_i \Phi(|a_i\rangle \langle a_i|) | b_j \rangle = \langle b_j | \Phi(\mathbb{1}_A) | b_j \rangle = \frac{d_A}{d_B} . \quad (21)$$

For  $p(a_i) = \text{Tr}(e^{-\alpha A})^{-1} \exp(-\alpha a_i)$  in (15), we then express the left-hand side of (20) as

$$\sum_{ij} \frac{\exp(-\alpha a_i)}{\text{Tr}(e^{-\alpha A})} p(b_j|a_i) \exp(\alpha a_i - \beta b_j) = \frac{1}{\text{Tr}(e^{-\alpha A})} \sum_j \exp(-\beta b_j) \frac{d_A}{d_B}. \quad (22)$$

The latter term is equal to the right-hand side of (20). ■

Due to (9), we have evaluated the sum (21) in a closed form. Hence, the claim (20) was immediately obtained. Using the result (20), we will further discuss the Jarzynski equality. We should emphasize a distinction of the sum (21) from the constraint (13), which takes the sum with respect to the final events. This point may be illustrated with the equiprobable distribution  $p(a_i, b_j) = (d_A d_B)^{-1}$ . Hence, we obtain the conditional probabilities  $p(b_j|a_i) = d_B^{-1}$  and the right-hand side of (21). In the context of quantum channels, such probabilities are realized in the case  $d_A \leq d_B$ . Let us consider an isometry  $V : \mathcal{H}_A \rightarrow \mathcal{H}_B$  such that the orthonormal set  $\{V|a_i\rangle\}$  is mutually unbiased with the basis  $\{|b_j\rangle\}$ . In other words, one gives  $|\langle b_j|V|a_i\rangle| = d_B^{-1/2}$ . The trace-preserving map  $\Phi$  is then defined by (2) with a unique Kraus operator  $K = V$ . In many cases of interest, the system dimensionality is not altered during a physical process, i.e.  $d_A = d_B$ .

Let us proceed to an extension of Jarzynski's equality [1, 6]. We assume that a thermally contacted system is acted upon by an external agent. This agent operates according to a specified protocol. Hence, the Hamiltonian of the system is time-dependent. The principal system is initially prepared in the state of thermal equilibrium with a heat reservoir. Following Tasaki [13], we will firstly assume that the reservoir temperature is also dependent on the time. The parameters  $\beta_0$  and  $\beta_1$  give the inverse temperature of the reservoir at the initial and final moments, respectively. Thus, the initial density matrix is

$$\omega_0(\beta_0) = Z_0(\beta_0)^{-1} e^{-\beta_0 H_0}, \quad (23)$$

in terms of the initial Hamiltonian  $H_0$  and the corresponding partition function  $Z_0(\beta_0) = \text{Tr}(e^{-\beta_0 H_0})$ . We further suppose that the transformation of states of the system is represented by quantum channel  $\Phi$  with the same input and output Hilbert space. In general, the final density matrix  $\Phi(\omega_0(\beta_0))$  will enough differ from the matrix

$$\omega_1(\beta_1) = Z_1(\beta_1)^{-1} e^{-\beta_1 H_1}, \quad (24)$$

corresponding to equilibrium at the final moment. Here, the partition function  $Z_1(\beta_1) = \text{Tr}(e^{-\beta_1 H_1})$  is expressed in terms of the final Hamiltonian  $H_1$ . By  $\{\varepsilon_m^{(0)}\}$  and  $\{\varepsilon_n^{(1)}\}$ , we respectively denote eigenvalues of the Hamiltonians  $H_0$  and  $H_1$ . Taking  $d_A = d_B$  in (9) implies that the map is unital. By obvious substitutions, the formula (20) then gives

$$\left\langle \exp\left(\beta_0 \varepsilon^{(0)} - \beta_1 \varepsilon^{(1)}\right) \right\rangle = \frac{Z_1(\beta_1)}{Z_0(\beta_0)}, \quad (25)$$

provided that the channel  $\Phi$  is unital. Assuming unitary evolution, the relation (25) has been derived by Tasaki [13]. So, we have extended an important formulation to unital quantum channels.

In the considered context, the term  $w_{nm} = \varepsilon_n^{(1)} - \varepsilon_m^{(0)}$  is treated as an external work performed on the principal system during a process. At constant temperature, i.e. when  $\beta_0 = \beta_1 = \beta$ , we therefore have

$$\left\langle \exp(-\beta w) \right\rangle = \exp(-\beta \Delta F), \quad (26)$$

since  $F_t(\beta) = -\beta^{-1} \ln Z_t(\beta)$  for  $t = 0, 1$ . The result (26) relates, on average, a non-equilibrium external work with the difference  $\Delta F = F_1 - F_0$  between the equilibrium free energies. This formula is the original Jarzynski equality [6, 7]. As a consequence, the basic inequality of thermodynamics can be obtained. Combining (26) with Jensen's inequality for convex function  $x \mapsto \exp(-\beta x)$  leads to

$$\exp(-\beta \langle w \rangle) \leq \exp(-\beta \Delta F). \quad (27)$$

As the function  $x \mapsto \exp(-\beta x)$  decreases with  $x$  for positive  $\beta$ , the formula (27) gives  $\langle w \rangle \geq \Delta F$ . Thus, total external work will, on average, exceed the difference between values of the equilibrium free energy at the final and initial moments. Basing on Jarzynski's equality, Tasaki also discussed some inequalities for the von Neumann entropy [13].

## B. Theorem of Tasaki–Crooks type for bistochastic map and its adjoint

Fluctuation theorems are typically used in studying stochastic processes. They are still the subject of active research [42–45]. Such theorems can be used for deriving information-theoretic results, for instance, Holevo's bound [17]. For

the Tasaki–Crooks fluctuation theorem, a development with unital quantum channels is reasoned as follows. For trace-preserving map  $\Phi$ , its adjoint  $\Phi^\dagger$  is unital. To make  $\Phi^\dagger$  trace-preserving, the map  $\Phi$  itself should be unital as well. For this reason, we focus an attention on bistochastic maps, i.e. on unital quantum channels. By  $\mathcal{H}$ , we denote the Hilbert space assigned to the principal system. Similarly to (19), we introduce the density matrix

$$\varrho_B(\alpha) := \text{Tr}(e^{-\alpha B})^{-1} e^{-\alpha B} . \quad (28)$$

Consider two processes obtained by applying the channel  $\Phi$  to the input  $\varrho_A$  and the channel  $\Phi^\dagger$  to the input  $\varrho_B$ . In each of the processes, we can ask for a probability that the difference  $(a_i - b_j)$  takes a certain value. It turns out that the two corresponding probabilities obey some relation. For a difference between eigenvalues of the Hamiltonians  $H_0$  and  $H_1$ , results of such a kind are usually referred to as the Tasaki–Crooks theorem [8, 18]. It will be convenient, however, to pose a statement in more abstract form. Let us put the notation. Imposing the restriction  $b_j - a_i = \Delta$ , we fix a difference between eigenvalues of the observables  $B$  and  $A$ . By  $P(b - a = \Delta | \Phi, \varrho_A)$ , we denote the probability of this event given that the channel  $\Phi$  represents an evolution of the input  $\varrho_A$ . We have the following result.

**Proposition 2** *Let the quantum channel  $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  be unital. For all real  $\Delta$ , the corresponding probabilities satisfy*

$$e^{-\alpha \Delta} \text{Tr}(e^{-\alpha A}) P(b - a = \Delta | \Phi, \varrho_A) = \text{Tr}(e^{-\alpha B}) P(a - b = -\Delta | \Phi^\dagger, \varrho_B) , \quad (29)$$

where the density matrices  $\varrho_A$  and  $\varrho_B$  are respectively defined by (19) and (28).

**Proof.** Assuming action of the channel  $\Phi^\dagger$ , we use the conditional probability of outcome  $a_i$  given that the input state was  $|b_j\rangle$ . Similarly to (12), this probability is written as

$$q(a_i | b_j) = \langle a_i | \Phi^\dagger(|b_j\rangle\langle b_j|) | a_i \rangle . \quad (30)$$

The derivation of (29) is based on a simple observation. Due to the representations (2) and (4), we have

$$\langle b | K_\mu | a \rangle \langle a | K_\mu^\dagger | b \rangle = \langle a | K_\mu^\dagger | b \rangle \langle b | K_\mu | a \rangle , \quad \langle b | \Phi(|a\rangle\langle a|) | b \rangle = \langle a | \Phi^\dagger(|b\rangle\langle b|) | a \rangle , \quad (31)$$

with arbitrary  $|a\rangle, |b\rangle \in \mathcal{H}$ . For all  $i$  and  $j$ , therefore, one satisfies

$$p(b_j | a_i) = q(a_i | b_j) . \quad (32)$$

The matrix (19) has eigenvalues  $p(a_i) = \text{Tr}(e^{-\alpha A})^{-1} \exp(-\alpha a_i)$ . Let  $\mathcal{N}(\Delta)$  be the set of ordered pairs  $(j, i)$  such that  $b_j - a_i = \Delta$ . By (32), the left-hand side of (29) is then rewritten as

$$\begin{aligned} \sum_{(j,i) \in \mathcal{N}(\Delta)} \exp(-\alpha b_j + \alpha a_i) \text{Tr}(e^{-\alpha A}) p(a_i) p(b_j | a_i) &= \sum_{(j,i) \in \mathcal{N}(\Delta)} \exp(-\alpha b_j) p(b_j | a_i) \\ &= \text{Tr}(e^{-\alpha B}) \sum_{(j,i) \in \mathcal{N}(\Delta)} q(b_j) q(a_i | b_j) . \end{aligned} \quad (33)$$

Here the numbers  $q(b_j) = \text{Tr}(e^{-\alpha B})^{-1} \exp(-\alpha b_j)$  are eigenvalues of  $\varrho_B$ . For all the ordered pairs  $(j, i) \in \mathcal{N}(\Delta)$ , we have  $a_i - b_j = -\Delta$ . Combining this with the definition (30), the right-hand side of (33) is equal to the right-hand side of (29). ■

We can also write some relation with the completely mixed state  $\rho_* = \mathbb{1}/d$ , where  $d = \dim(\mathcal{H})$  and  $p(a_i) = q(b_j) = 1/d$ . Repeating the above reasons, we obtain

$$P(b - a = \Delta | \Phi, \rho_*) = P(a - b = -\Delta | \Phi^\dagger, \rho_*) . \quad (34)$$

It seems that the matrices of the form (19) and the  $\rho_*$  are the only two forms, for which a closed relation between the two probabilities could be written.

The statement of Proposition 2 immediately leads to a theorem of Tasaki–Crooks type. Let us consider a thermally insulated system acted upon by an external field. As above, the operators  $H_0$  and  $H_1$  are respectively the initial and final Hamiltonians. For a unital quantum channel  $\Phi$ , we apply this channel itself to the equilibrium state (23) and the adjoint  $\Phi^\dagger$  to the equilibrium state (24). Differences  $w_{nm} = \varepsilon_n^{(1)} - \varepsilon_m^{(0)}$  are treated as possible values of an external work performed on the system during the former process. By obvious substitution, the formula (29) gives

$$e^{-\beta w} Z_0(\beta) P(\varepsilon^{(1)} - \varepsilon^{(0)} = w | \Phi, \omega_0) = Z_1(\beta) P(\varepsilon^{(0)} - \varepsilon^{(1)} = -w | \Phi^\dagger, \omega_1) , \quad (35)$$

with the inverse temperature  $\beta$  of heat reservoir. Using  $\Delta F = F_1 - F_0$ , the relation (35) can be rewritten in the form

$$\frac{P(\varepsilon^{(1)} - \varepsilon^{(0)} = w | \Phi, \omega_0)}{P(\varepsilon^{(0)} - \varepsilon^{(1)} = -w | \Phi^\dagger, \omega_1)} = \exp(\beta w - \beta \Delta F) , \quad (36)$$

when probabilities are non-zero for taken  $w$ . If the channel  $\Phi$  represents a unitary evolution, then its adjoint  $\Phi^\dagger$  represents the inverse unitary evolution. In the case of unitary transformations, the exact relation (35) with the two probabilities has been derived by Tasaki [13]. It is a quantum analog of previous Crooks' formulation [11]. In the literature, the above statement is often referred to as the Tasaki–Crooks fluctuation theorem [15, 18]. Thus, we have obtained an extension of the Tasaki–Crooks fluctuation theorem to unital quantum channels. Finally, we emphasize that use of adjoint maps in the formulation inevitably leads to unital channels. Indeed, the adjoint map is trace-preserving only for a bistochastic map.

### C. Notes on heat transfer between two systems

We now apply the above results to a heat transfer between two quantum systems. The following analysis is an extension of related results of the paper [13]. It is convenient, however, to put a derivation in a more abstract manner. Consider two systems with the Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. So, we write the initial density matrix of the combined system as

$$\varrho_{AB} := \varrho_A(\alpha) \otimes \varrho_B(\beta) = \text{Tr}(e^{-\alpha A})^{-1} \text{Tr}(e^{-\beta B})^{-1} e^{-\alpha A} \otimes e^{-\beta B} , \quad (37)$$

with  $A \in \mathcal{L}_{s.a.}(\mathcal{H}_A)$  and  $B \in \mathcal{L}_{s.a.}(\mathcal{H}_B)$ . Here, we used the matrices (19) and (28), but the latter with  $\beta$  instead of  $\alpha$ . We also assume that a state change of the combined system is represented by the channel  $\Psi$ , with the input space  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  and the output space  $\mathcal{H}_C$ . Let us take quantum channels that satisfy condition of the form (9), namely

$$\Psi(\mathbb{1}_{AB}) = \frac{d_A d_B}{d_C} \mathbb{1}_C , \quad (38)$$

where  $\mathbb{1}_{AB} = \mathbb{1}_A \otimes \mathbb{1}_B$  and  $\mathbb{1}_C$  are the corresponding identities. The operator  $\alpha A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \beta B$  is Hermitian for real  $\alpha$  and  $\beta$ . It has eigenvalues  $\alpha a_i + \beta b_j$  and eigenstates  $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$ . In terms of this operator, the matrix (37) reads

$$\varrho_{AB} := \text{Tr}\left(\exp(-\alpha A \otimes \mathbb{1}_B - \mathbb{1}_A \otimes \beta B)\right)^{-1} \exp(-\alpha A \otimes \mathbb{1}_B - \mathbb{1}_A \otimes \beta B) , \quad (39)$$

since the summands  $\alpha A \otimes \mathbb{1}_B$  and  $\mathbb{1}_A \otimes \beta B$  commute. For arbitrary  $C \in \mathcal{L}_{s.a.}(\mathcal{H}_C)$ , the following conclusion can be written as a variety of (20). If the quantum channel  $\Psi : \mathcal{L}(\mathcal{H}_{AB}) \rightarrow \mathcal{L}(\mathcal{H}_C)$  satisfies (38) and the input is given by (37), then

$$\left\langle \exp(\alpha a + \beta b - c) \right\rangle = \frac{d_A d_B}{d_C} \frac{\text{Tr}(e^{-C})}{\text{Tr}(e^{-\alpha A}) \text{Tr}(e^{-\beta B})} . \quad (40)$$

In the case  $\mathcal{H}_C = \mathcal{H}_{AB}$  and  $C = \alpha A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \beta B$ , the formula (40) becomes

$$\left\langle \exp(\alpha(a - a') + \beta(b - b')) \right\rangle = 1 , \quad (41)$$

where the set  $\{\alpha a'_i + \beta b'_j\}$  denotes the spectrum of  $C$ . The condition (38) is reduced here to  $\Psi(\mathbb{1}_{AB}) = \mathbb{1}_{AB}$ .

We now consider the following question. Each of two separated systems is initially prepared in equilibrium at the inverse temperatures  $\beta_0$  and  $\beta_1$ , respectively. Their density matrices are therefore written as (23) and (24), whence the product  $\omega_0(\beta_0) \otimes \omega_1(\beta_1)$  is the input total state. Suppose that the systems further interact via unital quantum channel  $\Psi$ . By obvious substitutions into (41), for the described process we obtain

$$\left\langle \exp\left(\beta_0(\varepsilon^{(0)} - \varepsilon'^{(0)}) + \beta_1(\varepsilon^{(1)} - \varepsilon'^{(1)})\right) \right\rangle = 1 . \quad (42)$$

Assuming unitary evolution, this result has been given in [13]. Thus, we have extended the previous result to unital quantum channels. In the paper [13], the relation (42) is presented with a certain physical interpretation. Let us consider the quantity

$$\Delta S := \left\langle \beta_0(\varepsilon'^{(0)} - \varepsilon^{(0)}) + \beta_1(\varepsilon'^{(1)} - \varepsilon^{(1)}) \right\rangle . \quad (43)$$

The terms  $\langle \varepsilon'^{(0)} - \varepsilon^{(0)} \rangle$  and  $\langle \varepsilon'^{(1)} - \varepsilon^{(1)} \rangle$  give, on average, the change of self-energy of the corresponding systems. Suppose that changes in the inverse temperatures of the systems are sufficiently small and a contribution of interaction energy to the total entropy is negligible. In such a situation, the quantity  $\Delta S$  estimates an averaged change of the total entropy [13]. Combining (42) with Jensen's inequality for convex function  $x \mapsto \exp(-x)$ , we obtain

$$\exp(-\Delta S) \leq 1, \quad \Delta S \geq 0. \quad (44)$$

Thus, we have arrived at the well-known inequality of thermodynamics. An extension of Clausius' inequality to arbitrary non-equilibrium processes beyond linear response was obtained in [46]. In the paper [13], Tasaki also discussed a way for constructing more detailed bounds on  $\Delta S$  from below. We only emphasize here that this can be achieved with unital quantum channels.

Thus, we have shown that Jarzynski's equality and many related results remain valid in the case, when the evolution of a quantum system is represented by unital quantum channels. In this regard, we do not need in the unitarity assumption, which is much more restrictive. Let us discuss briefly a significance of unital channels and their distinctions from unitary ones. Unitary channels form a very special class of quantum channels. If the given quantum channel is invertible for all inputs, then it is unitary with necessity [47]. Here, we do not mean the invertibility for the prescribed input, as treated in the data processing inequality [37]. With respect to the map composition, the set of all unitary channels has a structure of group. To consider dynamics of open quantum systems, we have to left out invertibility. Assuming the most general form of state changes, we should focus an attention on dynamical semigroups. Indeed, the composition of two trace-preserving maps is also trace-preserving. It is not insignificant that similar observation pertains to unitality. Namely, the composition of two unital maps is unital as well. Thus, bistochastic maps form the set with a semigroup structure. For all such maps, the Jarzynski equality and the Tasaki–Crooks fluctuation theorem are still valid. The simplest examples of unital and non-unitary channels are the phase damping and depolarizing qubit channels. Both are examples of subtle and important quantum processes. For example, key effects in the Schrödinger cat-atom system may be modeled as phase damping [37]. The depolarizing channel represents a typical case of decohering qubit [47]. Another reason to consider unital channels was pointed out in connection with the Tasaki–Crooks fluctuation theorem. The role of unitality in statistical physics of small quantum systems deserves further investigations.

#### IV. JARZYNSKI–SAGAWA–UEDA RELATIONS WITH UNITAL QUANTUM CHANNELS

In this section, we develop some of the above results in the case, when the agent makes measurement followed by a feedback. First, error-free feedback control is considered. Second, we analyze the case, in which classical errors occur in the measurement process.

##### A. Error-free feedback control

We assume that the agent performs a quantum measurement and further acts according to the measurement outcome. For classical systems, this topic has been considered by Sagawa and Ueda [28, 29]. For quantum systems, relations of such a kind were examined by Morikuni and Tasaki [19]. Let us recall required material on quantum measurements. In general, the quantum measurement is posed as a set  $\{N_\mu\}$  of measurement operators, acting on the space of the measured system [37]. If the pre-measurement state is described by  $\rho$ , then the probability of  $\mu$ -th outcome is  $\text{Tr}(N_\mu^\dagger N_\mu \rho)$ . The corresponding post-measurement state is described by density matrix

$$\rho'_\mu = \text{Tr}(N_\mu^\dagger N_\mu \rho)^{-1} N_\mu \rho N_\mu^\dagger. \quad (45)$$

Note that number of measurement outcomes can arbitrarily exceed dimensionality of the Hilbert space. This possibility is crucial for many quantum protocols [37]. The set of measurement operators satisfies the completeness relation

$$\sum_\mu N_\mu^\dagger N_\mu = \mathbb{1}. \quad (46)$$

When the measurement operators are mutually orthogonal projectors, the above scheme obviously leads to traditional projective measurements. We will also say about quantum measurements that fulfill the condition

$$\sum_\mu N_\mu N_\mu^\dagger = \mathbb{1}. \quad (47)$$



The projective measurements all satisfy the condition (47). Moreover, for Hermitian measurement operators this condition coincides with the completeness relation (46).

Let us pose formally the protocol with error-free feedback control. For the sake of simplicity, we will assume the same input and output space for all adopted operations. Further, we use an appropriate number of observables  $B_\mu \in \mathcal{L}_{s.a.}(\mathcal{H})$ , each with its eigenbasis  $\{|b_j^{(\mu)}\rangle\}$ . We also define the corresponding density matrices  $\varrho_B^{(\mu)}(\alpha)$  by the formula (28) with  $B_\mu$  instead of  $B$ . We shall consider the following procedure.

- (i) At the first stage, the agent applies quantum channel  $\Phi$  to the input  $\varrho_A$  given by (19).
- (ii) At the second stage, one performs the quantum measurement on the output  $\Phi(\varrho_A)$ . For  $\mu$ -th outcome, its probability is  $p(\mu) = \text{Tr}(\mathbf{N}_\mu^\dagger \mathbf{N}_\mu \Phi(\varrho_A))$  and the post-measurement state is  $p(\mu)^{-1} \mathbf{N}_\mu \Phi(\varrho_A) \mathbf{N}_\mu^\dagger$ .
- (iii) At the third stage, the agent applies the prescribed quantum channel  $\Psi_\mu$  to  $\mu$ -th post-measurement state given that  $\mu$ -th outcome occurs.
- (iv) At the fourth stage, one measures the observable  $B_\mu$  on the third-stage output  $p(\mu)^{-1} \Psi_\mu(\mathbf{N}_\mu \Phi(\varrho_A) \mathbf{N}_\mu^\dagger)$ . With this pre-measurement state, the outcome  $b_j^{(\mu)}$  is obtained with the probability

$$p(\mu)^{-1} \langle b_j^{(\mu)} | \Psi_\mu(\mathbf{N}_\mu \Phi(\varrho_A) \mathbf{N}_\mu^\dagger) | b_j^{(\mu)} \rangle . \quad (48)$$

Multiplying (48) by  $p(\mu)$ , i.e. by  $\mu$ -th outcome probability, we obtain the probability of the outcome  $b_j^{(\mu)}$  for the input (19). The latter probability can be represented as the sum

$$\sum_i p(a_i, b_j^{(\mu)}) = \sum_i p(a_i) p(b_j^{(\mu)} | a_i) . \quad (49)$$

Here, we introduce the conditional probability

$$p(b_j^{(\mu)} | a_i) = \langle b_j^{(\mu)} | \Psi_\mu(\mathbf{N}_\mu \Phi(|a_i\rangle\langle a_i|) \mathbf{N}_\mu^\dagger) | b_j^{(\mu)} \rangle , \quad (50)$$

given that the input was  $|a_i\rangle$ . These probabilities satisfy the required condition

$$\sum_{j\mu} p(b_j^{(\mu)} | a_i) = \sum_\mu \text{Tr}(\mathbf{N}_\mu^\dagger \mathbf{N}_\mu \Phi(|a_i\rangle\langle a_i|)) = 1 . \quad (51)$$

We have used the preservation and the cyclic property of the trace and the completeness relation (46). For clarity, all the labels of involved variables will be explicitly indicated between the angular brackets.

**Proposition 3** *Let the above protocol be applied to the input (19), and let the quantum channel  $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  be unital. For arbitrary quantum measurement  $\{\mathbf{N}_\mu\}$  and quantum channels  $\Psi_\mu$ , we have*

$$\left\langle \frac{\text{Tr}(e^{-\alpha A})}{\text{Tr}(e^{-\alpha B_\mu})} \exp(\alpha a_i - \alpha b_j^{(\mu)}) \right\rangle = \sum_\mu \text{Tr}(\varrho_B^{(\mu)}(\alpha) \Psi_\mu(\mathbf{N}_\mu \mathbf{N}_\mu^\dagger)) . \quad (52)$$

**Proof.** Since the channel  $\Phi$  is unital and  $\langle b_j^{(\mu)} | \exp(-\alpha b_j^{(\mu)}) = \langle b_j^{(\mu)} | \exp(-\alpha B_\mu)$ , we first write

$$\begin{aligned} \frac{\text{Tr}(e^{-\alpha A})}{\text{Tr}(e^{-\alpha B_\mu})} \sum_i p(a_i) p(b_j^{(\mu)} | a_i) \exp(\alpha a_i - \alpha b_j^{(\mu)}) &= \frac{\exp(-\alpha b_j^{(\mu)})}{\text{Tr}(e^{-\alpha B_\mu})} \langle b_j^{(\mu)} | \Psi_\mu(\mathbf{N}_\mu \Phi(\mathbb{1}) \mathbf{N}_\mu^\dagger) | b_j^{(\mu)} \rangle \\ &= \langle b_j^{(\mu)} | \varrho_B^{(\mu)}(\alpha) \Psi_\mu(\mathbf{N}_\mu \mathbf{N}_\mu^\dagger) | b_j^{(\mu)} \rangle . \end{aligned} \quad (53)$$

Summing the right-hand side of (53) with respect to  $j$ , we get  $\text{Tr}(\varrho_B^{(\mu)}(\alpha) \Psi_\mu(\mathbf{N}_\mu \mathbf{N}_\mu^\dagger))$ . The latter leads to (52) after further summing with respect to  $\mu$ . ■

The statement of Proposition 3 is written in a general form. We now apply this result to a thermally insulated system, which is operated by the agent with feedback control. For each of the possible ways, we introduce the corresponding Hamiltonian  $H_\mu$  and equilibrium state

$$\omega_\mu(\beta) = Z_\mu(\beta)^{-1} e^{-\beta H_\mu} , \quad Z_\mu(\beta) = \text{Tr}(e^{-\beta H_\mu}) . \quad (54)$$

The protocol is applied to the input  $\omega_0(\beta)$ , whereas the averaging is taken over the final states  $\omega_\mu(\beta)$  with  $\mu \neq 0$ . Let us introduce the associated work  $w_{nm}^{(\mu)} = \varepsilon_n^{(\mu)} - \varepsilon_m^{(0)}$ . By obvious substitution, we rewrite the equality (52) as

$$\left\langle \exp\left(-\beta w_{nm}^{(\mu)} + \beta(F_\mu - F_0)\right) \right\rangle = \gamma, \quad (55)$$

where  $F_\mu(\beta) = -\beta^{-1} \ln Z_\mu(\beta)$  and the parameter  $\gamma$  is defined by

$$\gamma := \sum_\mu \text{Tr}\left(\omega_\mu(\beta) \Psi_\mu(\mathbf{N}_\mu \mathbf{N}_\mu^\dagger)\right). \quad (56)$$

The formula (55) is a quantum counterpart of one of the results originally formulated in [28]. With unitary transformations of quantum states, the relation (55) was derived in [19]. For two projective measurements and any trace-preserving map between them, a similar relation was considered in [17]. Thus, we have extended an important non-equilibrium equality to the case, when the unital channel  $\Phi$  acts at the stage (i) and arbitrary channels  $\Psi_\mu$  act at the stage (iii). Let us discuss a particular case without any feedback. Here, the same channel  $\Psi$  is applied for all the outcomes and final states are always compared with  $\omega_1(\beta)$ . If the channel  $\Psi$  is unital and the POVM  $\{\mathbf{N}_\mu\}$  obeys (47), then we have  $\gamma = \text{Tr}(\omega_1(\beta)) = 1$ . It is Jarzynski's equality (26) with an intermediate quantum measurement.

### B. Feedback control with classical errors

The above result can be modified to the case, when measurement outcomes are registered with some randomness. By probabilities  $r(\nu|\mu)$ , we represent purely classical nature of errors at the stage (ii). The quantity  $r(\nu|\mu)$  is the conditional probability of mis-interpretation of actual  $\mu$ -th outcome as registered  $\nu$ -th one. Recall the concept of mutual information. The pointwise mutual information is defined as [48]

$$I_{\mu\nu} := \ln \frac{p(\mu, \nu)}{p(\mu)p(\nu)} = \ln \frac{r(\mu|\nu)}{p(\mu)} = \ln \frac{r(\nu|\mu)}{p(\nu)}, \quad (57)$$

where we used Bayes' rule. This quantity can take positive or negative values, vanishing for  $p(\mu, \nu) = p(\mu)p(\nu)$ . Averaging (57) with the joint probability distribution, we obtain the mutual information

$$\langle I_{\mu\nu} \rangle = \sum_{\mu\nu} p(\mu, \nu) \ln \frac{p(\mu, \nu)}{p(\mu)p(\nu)}. \quad (58)$$

The mutual information is extensively treated in information theory [48].

In the case of feedback control with classical errors, we should average with the joint probability distribution

$$p(a_i, \mu, b_j^{(\nu)}) = p(a_i) p(\mu, b_j^{(\nu)} | a_i), \quad (59)$$

in which the latter conditional probability is written as

$$p(\mu, b_j^{(\nu)} | a_i) = r(\nu|\mu) \langle b_j^{(\nu)} | \Psi_\nu(\mathbf{N}_\mu \Phi(|a_i\rangle\langle a_i|) \mathbf{N}_\mu^\dagger) | b_j^{(\nu)} \rangle. \quad (60)$$

For error-free feedback control, we have  $r(\nu|\mu) = \delta_{\nu\mu}$ . Here, the right-hand side of (60) is non-zero only for  $\nu = \mu$ , when it is reduced to the right-hand side of (50). Similarly to (51), by means of  $\sum_\nu r(\nu|\mu) = 1$  we also obtain

$$\sum_{j\mu\nu} p(\mu, b_j^{(\nu)} | a_i) = 1. \quad (61)$$

**Proposition 4** *Let the above protocol be applied to the input (19), and let classical errors at the stage (ii) be represented by the conditional probability  $r(\nu|\mu)$ . If the quantum channel  $\Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  is unital, then*

$$\left\langle \frac{\text{Tr}(e^{-\alpha \mathbf{A}})}{\text{Tr}(e^{-\alpha \mathbf{B}_\nu})} \exp(\alpha a_i - \alpha b_j^{(\nu)}) \right\rangle = \sum_{\mu\nu} r(\nu|\mu) \text{Tr}\left(\mathcal{Q}_B^{(\nu)}(\alpha) \Psi_\nu(\mathbf{N}_\mu \mathbf{N}_\mu^\dagger)\right). \quad (62)$$

*If the quantum channels  $\Psi_\mu : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$  are also all unital and the POVM  $\{\mathbf{N}_\mu\}$  obeys (47) then*

$$\left\langle \frac{\text{Tr}(e^{-\alpha \mathbf{A}})}{\text{Tr}(e^{-\alpha \mathbf{B}_\nu})} \exp(\alpha a_i - \alpha b_j^{(\nu)} - I_{\nu\mu}) \right\rangle = 1. \quad (63)$$

**Proof.** Similarly to the proof of Proposition 3, we first obtain the relation

$$\frac{\text{Tr}(e^{-\alpha A})}{\text{Tr}(e^{-\alpha B_\nu})} \sum_{ij} p(a_i) p(\mu, b_j^{(\nu)} | a_i) \exp(\alpha a_i - \alpha b_j^{(\nu)}) = r(\nu | \mu) \text{Tr} \left( \boldsymbol{\varrho}_B^{(\nu)}(\alpha) \Psi_\nu(\mathbf{N}_\mu \mathbf{N}_\mu^\dagger) \right), \quad (64)$$

provided that the channel  $\Phi$  is unital. After summing with respect to  $\mu$  and  $\nu$ , we get the first claim (62). Multiplying (64) by  $\exp(-I_{\nu\mu}) = p(\nu)/r(\nu|\mu)$  and further summing with respect to  $\mu$ , one yields

$$\frac{\text{Tr}(e^{-\alpha A})}{\text{Tr}(e^{-\alpha B_\nu})} \sum_{ij\mu} p(a_i) p(\mu, b_j^{(\nu)} | a_i) \exp(\alpha a_i - \alpha b_j^{(\nu)} - I_{\nu\mu}) = p(\nu) \text{Tr} \left( \boldsymbol{\varrho}_B^{(\nu)}(\alpha) \Psi_\nu(\mathbb{1}) \right), \quad (65)$$

under the condition (47). If  $\Psi_\nu(\mathbb{1}) = \mathbb{1}$  for all  $\nu$ , summing of (65) with respect to  $\nu$  finally gives  $\sum_\nu p(\nu) = 1$ . ■

It is essential for (62) and (63) that corresponding quantum operations be unital. The proof of the result (62) assumes this property only for the channel at the first stage (i). The proof of the result (63) has also assumed this property for the measurement at the stage (ii) and for all the channels at the stage (iii). Let us proceed to a thermally insulated system, which is operated by the agent with feedback control. In the considered case, the formula (62) gives

$$\left\langle \exp \left( -\beta w_{nm}^{(\nu)} + \beta(F_\nu - F_0) \right) \right\rangle = \tilde{\gamma}, \quad (66)$$

where the parameter  $\tilde{\gamma}$  is written as

$$\tilde{\gamma} := \sum_{\mu\nu} r(\nu | \mu) \text{Tr} \left( \boldsymbol{\omega}_\nu(\beta) \Psi_\nu(\mathbf{N}_\mu \mathbf{N}_\mu^\dagger) \right). \quad (67)$$

Replacing  $\gamma$  with  $\tilde{\gamma}$ , the result (66) is completely similar to (55). For error-free feedback control, the term  $\tilde{\gamma}$  becomes  $\gamma$  due to  $r(\nu | \mu) = \delta_{\nu\mu}$ . The formulas (56) and (67) express the parameters in a closed form. Notably, these parameters are experimentally measurable quantities. The original Sagawa–Ueda formulation with feedback control is classical. It has been tested experimentally [27]. The quantum results (55) and (66) may be used in the context of future experiments. They hold when the channel  $\Phi$  is unital. If the quantum operations at the stages (ii) and (iii) are unital as well, then

$$\left\langle \exp \left( -\beta w_{nm}^{(\nu)} + \beta(F_\nu - F_0) - I_{\nu\mu} \right) \right\rangle = 1. \quad (68)$$

This is a quantum counterpart of one of the results obtained by Sagawa and Ueda [28]. In the paper [19], the relation (68) has been presented within a unitary evolution under assumption that the measurement obeys (47). The formula (66) has also been given in [19] with unitary transformations of the states. Thus, we have extended quantum Jarzynski–Sagawa–Ueda relations to unital quantum channels.

## V. CONCLUSIONS

We have considered the Jarzynski equality and related fluctuation results from the viewpoint of quantum operation techniques. The formalism of quantum operations was developed to describe dynamics of open quantum systems. Since study of open systems is one of key aims of statistical physics, the language of quantum operations may offer new insights. In the paper, we have considered some advances of the mentioned viewpoint. It is very essential that representing quantum channels be unital. Assuming this property, the Jarzynski equality is still valid. A fluctuation theorem of Tasaki–Crooks type has been formulated for bistochastic map and its adjoint. With unital quantum channels, we also apply the formalism to the problem of heat transfer between two quantum systems. Some equalities with unital channels have also been derived in the case of feedback control. Error-free feedback and feedback with classical errors are both considered. Hence, quantum Jarzynski–Sagawa–Ueda relations have been generalized to unital quantum channels. The obtained expressions may be useful in experimental tests. Thus, the formalism of quantum operations provides a suitable framework for studying non-equilibrium relations in open quantum systems. The described approach is interesting in own rights as well as from the viewpoint of future experimental validation of the quantum non-equilibrium results. First, novel aspects of the problem could be analyzed in terms of quantum stochastic maps. In particular, this approach is well adapted to describe noise in open quantum systems. Indeed, for existing or future proposals, we should estimate a degree of environmental noise and its influence on experimental results. Second, the formalism of quantum operations is now common language to represent state transformations in quantum information processing. In principle, practical achievements in quickly growing area of quantum information may be used in experimental tests of non-equilibrium relations in quantum systems.

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*Note added.* After the present paper was submitted I learned about the recent work [49], in which the significance of unitality is emphasized as well. In the work [49], the authors formulate a general fluctuation theorem and further show that some previous results follow from this theorem. My formulations and derivation methods are different from and, in certain respects, complementary to those given in [49]. The authors of [49] also describe results of a related experiment with superconducting flux qubits.

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